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Approximate Solutions of Multiobjective Optimization Problems

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1 Introduction and Preliminaries

In this paper, we employ the *limiting subdifferential* and the *Mordukhovich normal cone* (cf. [7]) to examine approximate Pareto solutions of a multiobjective optimization problem. More precisely, we establish Fritz-John type necessary conditions for ϵ-(*weakly*) *Pareto solutions* and ϵ-*quasi*-(*weakly*) *Pareto solutions* of a multiobjective optimization problem involving *nonsmooth/nonconvex* functions.

With the help of *generalized convex functions* defined in terms of the limiting subdifferential and the Mordukhovich normal cone, the obtained necessary conditions for approximate Pareto solutions of the considered problem become *sufficient* ones. In this way, we are able to explore completely *duality relations* for approximate Pareto solutions between multiobjective optimization problems such as strong duality and converse duality.

Throughout the paper we use the standard notation of variational analysis; see e.g., [7]. Unless otherwise specified, all spaces under consideration are *Asplund* spaces whose norms are always denoted by $\|\cdot\|$. The canonical pairing between space X and its dual X^* is denoted by $\langle\cdot,\cdot\rangle$. The symbol B_X stands for the closed unit ball in X. As usual, the *polar cone* of $\Omega\subset X$ is the set

$$\Omega^\circ := \{x^*\in X^* \mid \langle x^*, x\rangle \le 0 \quad \forall x\in\Omega\}. \tag{1.1}$$

Also, we denote by $\mathbb{R}^m_+$ the nonnegative orthant of $\mathbb{R}^m$, where $m\in\mathbb{N}:=\{1,2,\ldots\}$.

Given a set-valued mapping $F\colon X \rightrightarrows X^*$ between X and its dual X^*, we denote by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \Big|\ \exists \text{ sequences } x_n \to \bar{x} \text{ and } x_n^* \xrightarrow{w^*} x^* \\ \text{with } x_n^* \in F(x_n) \text{ for all } n \in \mathbb{N}\Big\}$$

the *sequential Painlevé-Kuratowski upper/outer limit* of F as $x \to \bar{x}$. Here the symbol $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of X^*.

A set $\Omega \subset X$ is called *closed around* $\bar{x} \in \Omega$ if there is a neighborhood U of $\bar{x}$ such that $\Omega \cap \operatorname{cl} U$ is closed. We say that Ω is *locally closed* if Ω is closed around x for every $x \in \Omega$. Let $\Omega \subset X$ be closed around $\bar{x} \in \Omega$.

The *Fréchet normal cone* to Ω at $\bar{x} \in \Omega$ is defined by

$$\widehat{N}(\bar{x};\Omega) := \Big\{x^* \in X^* \Big| \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le 0\Big\}, \tag{1.2}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. If $x \notin \Omega$, we put $\widehat{N}(x;\Omega) := \emptyset$.

The *limiting/Mordukhovich normal cone* $N(\bar{x};\Omega)$ to Ω at $\bar{x} \in \Omega$ is obtained from Fréchet normal cones by taking the sequential Painlevé-Kuratowski upper limits as:

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x;\Omega). \tag{1.3}$$

If $x \notin \Omega$, we put $N(x;\Omega) := \emptyset$.

For an extended real-valued function $\varphi : X \to \overline{\mathbb{R}} := [-\infty, \infty]$, we set

$$\operatorname{dom}\varphi := \{x \in X \mid \varphi(x) < \infty\}, \quad \operatorname{epi}\varphi := \{(x,\mu) \in X \times \mathbb{R} \mid \mu \ge \varphi(x)\}.$$

The *limiting/Mordukhovich subdifferential* of φ at $\bar{x} \in X$ with $|\varphi(\bar{x})| < \infty$ is defined by

$$\partial\varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \operatorname{epi}\varphi)\}. \tag{1.4}$$

If $|\varphi(\bar{x})| = \infty$, then one puts $\partial\varphi(\bar{x}) := \emptyset$.

Considering the indicator function $\delta(\cdot;\Omega)$ defined by $\delta(x;\Omega) := 0$ for $x \in \Omega$ and by $\delta(x;\Omega) := \infty$ otherwise, we have (see [7, Proposition 1.79]):

$$N(\bar{x};\Omega) = \partial\delta(\bar{x};\Omega) \quad \forall \bar{x} \in \Omega. \tag{1.5}$$

The *nonsmooth version of Fermat's rule* (see, e.g., [7, Proposition 1.114]) is formulated as follows: If $\bar{x}$ is a *local minimizer* for φ, then

$$0 \in \partial\varphi(\bar{x}). \tag{1.6}$$

For a function φ locally Lipschitz at $\bar{x}$ with modulus $\ell > 0$, it holds that (see [7, Corollary 1.81])

$$||x^*|| \leq \ell \quad \forall x^* \in \partial\varphi(\bar{x}). \tag{1.7}$$

2 Optimality Conditions for Approximate Solutions

This section is devoted to presenting optimality conditions for approximate solutions in multiobjective optimization prolems. Let Ω be a nonempty closed subset of X, and let $K := \{1, 2, ..., m\}$, and $I := \{1, 2, ..., p\}$ be index sets. Suppose that $f := (f_k),\ k \in K$, and $g := (g_i),\ i \in I$ are vector functions with locally Lipschitz components defined on X.

We focus on the following constrained multiobjective optimization problem (P):

$$\min\nolimits_{\mathbb{R}^m_+} \{f(x) \,|\, x \in C\}, \tag{2.8}$$

where C is the feasible set given by

$$C := \{x \in \Omega \mid g_i(x) \leq 0,\ i \in I\}. \tag{2.9}$$

Definition 2.1 ([5, 6]) Let $\epsilon := (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{R}^m_+$.
(i) We say that $\bar{x} \in C$ is an ϵ-*Pareto solution* of problem (2.8) iff there is no $x \in C$ such that

$$f_k(x) + \epsilon_k \leq f_k(\bar{x}), \quad k \in K \tag{2.10}$$

with at least one strict inequality.

(ii) A point $\bar{x} \in C$ is called an *ϵ-quasi-Pareto solution* of problem (2.8) iff there is no $x \in C$ such that

$$f_k(x) + \epsilon_k ||x - \bar{x}|| \leq f_k(\bar{x}), \quad k \in K \tag{2.11}$$

with at least one strict inequality.

If all the inequalities in (2.10) (resp., (2.11)) are strict, then one has the definition for *ϵ-weakly Pareto solution* (resp., *ϵ-quasi-weakly Pareto solution*) of problem (2.8). We denote the set of ϵ-Pareto solutions (resp., ϵ-weakly Pareto solutions, ϵ-quasi-Pareto solutions, and ϵ-quasi-weakly Pareto solutions) of problem (2.8) by ϵ-$\mathcal{S}(P)$ (resp., ϵ-$\mathcal{S}^w(P)$, ϵ-quasi-$\mathcal{S}(P)$, and ϵ-quasi-$\mathcal{S}^w(P)$). Note that *we always assume hereafter* that $\epsilon \in \mathbb{R}^m_+ \setminus \{0\}$.

To simplify the statements concerning problem (2.8), for fixed $\bar{x} \in X$ and $\epsilon \in \mathbb{R}^m_+ \setminus \{0\}$ we define (cf. [3]) a real-valued function ψ on X as follows:

$$\psi(x) := \max_{k \in K, i \in I} \{f_k(x) - f_k(\bar{x}) + \epsilon_k,\ g_i(x)\}, \quad x \in X. \tag{2.12}$$

Theorem 2.1 *Let $\bar{x} \in \epsilon$-$\mathcal{S}^w(P)$. For any $\nu > 0$, there exist $x_\nu \in \Omega$ and $\lambda_k \geq 0$, $k \in K$, $\mu_i \geq 0$, $i \in I$ with $\sum_{k \in K} \lambda_k + \sum_{i \in I} \mu_i = 1$, such that $||x_\nu - \bar{x}|| \leq \nu$ and*

$$\begin{aligned}
&0 \in \sum_{k \in K} \lambda_k \partial f_k(x_\nu) + \sum_{i \in I} \mu_i \partial g_i(x_\nu) + \frac{\max_{k \in K}\{\epsilon_k\}}{\nu} B_{X^*} + N(x_\nu; \Omega),\\
&\lambda_k \big[f_k(x_\nu) - f_k(\bar{x}) + \epsilon_k - \psi(x_\nu)\big] = 0, \quad k \in K,\\
&\mu_i \big[g_i(x_\nu) - \psi(x_\nu)\big] = 0, \quad i \in I,
\end{aligned}$$

where the function ψ was defined in (2.12).

The forthcoming theorem presents a Fritz-John type necessary condition for ϵ-quasi-(weakly) Pareto solutions of problem (2.8) with the help of Ekeland Variational Principle [2].

Theorem 2.2 *Let $\bar{x} \in \epsilon$-quasi-$\mathcal{S}^w(P)$. Then there exist $\lambda_k \geq 0$, $k \in K$, and $\mu_i \geq 0$, $i \in I$ with $\sum_{k\in K} \lambda_k + \sum_{i\in I} \mu_i = 1$, such that*

$$0 \in \sum_{k\in K} \lambda_k \partial f_k(\bar{x}) + \sum_{i\in I} \mu_i \partial g_i(\bar{x}) + \sum_{k\in K} \lambda_k \epsilon_k B_{X^*} + N(\bar{x};\Omega), \tag{2.13}$$

$$\mu_i g_i(\bar{x}) = 0, \quad i \in I.$$

Remark 2.1 According to Theorem 2.2, if $\bar{x}$ is an ϵ-quasi-(weakly) Pareto solution of problem (2.8), then the approximate (KKT) condition defined above is guaranteed by the following *constraint qualification* (CQ) due to [1](for special cases, one can see [4, 7, 8]): One says that condition (CQ) is satisfied at $\bar{x} \in C$ if there do not exist $\mu_i \geq 0, i \in I(\bar{x})$ not all zero, such that

$$0 \in \sum_{i\in I(\bar{x})} \mu_i \partial g_i(\bar{x}) + N(\bar{x};\Omega), \tag{2.14}$$

where $I(\bar{x}) := \{i \in I \mid g_i(\bar{x}) = 0\}$.

Theorem 2.3 *Let $\bar{x} \in C$ satisfy the ϵ-approximate (KKT) condition.*

(i) *If f and g are generalized convex on Ω at $\bar{x}$, then $\bar{x} \in \epsilon$-quasi-$\mathcal{S}^w(P)$.*

(ii) *If f is strictly generalized convex and g is generalized convex on Ω at $\bar{x}$, then $\bar{x} \in \epsilon$-quasi-$\mathcal{S}(P)$.*

3 Duality for Approximate Solutions

For $z \in X$, $\lambda := (\lambda_k)$, $\lambda_k \geq 0$, $k \in K$, and $\mu := (\mu_i)$, $\mu_i \geq 0$, $i \in I$, let us denote a vector Lagrangian function L by

$$L(z, \lambda, \mu) := f(z) + \langle \mu, g(z) \rangle e,$$

where $e := (1, \ldots, 1) \in \mathbb{R}^m$. In connection with the constrained multiobjective optimization problem (P) formulated in (2.8) and a given $\epsilon := (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{R}^m_+ \setminus \{0\}$, we consider a multiobjective dual problem in the following form (D):

$$\max\nolimits_{\mathbb{R}^m_+} \big\{ L(z, \lambda, \mu) \,|\, (z, \lambda, \mu) \in C_D \big\}. \tag{3.15}$$

Here the feasible set C_D is defined by

$$C_D := \Big\{(z,\lambda,\mu) \in \Omega \times (\mathbb{R}^m_+ \setminus \{0\}) \times \mathbb{R}^p_+ \mid 0 \in \sum_{k\in K} \lambda_k \partial f_k(z) + \sum_{i\in I} \mu_i \partial g_i(z) + \sum_{k\in K} \lambda_k \epsilon_k B_{X^*} + N(z;\Omega),\ \sum_{k\in K} \lambda_k = 1\Big\}. \tag{3.16}$$

Definition 3.1 Let $L := (L_1, \ldots, L_m)$, and let $\epsilon := (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{R}^m_+ \setminus \{0\}$. We say that $(\bar{z}, \bar{\lambda}, \bar{\mu}) \in C_D$ is an *ϵ-quasi-Pareto solution* of problem (3.15) iff there is no $(z,\lambda,\mu) \in C_D$ such that

$$L_k(z,\lambda,\mu) \geq L_k(\bar{z},\bar{\lambda},\bar{\mu}) + \epsilon_k ||(\bar{z},\bar{\lambda},\bar{\mu}) - (z,\lambda,\mu)||, \quad k \in K \tag{3.17}$$

with at least one strict inequality.

If all the inequalities in (3.17) are strict, then one has the definition for *ϵ-quasi-weakly Pareto solution* of problem (3.15). Also, the set of ϵ-quasi-Pareto solutions (resp., ϵ-quasi-weakly Pareto solutions) of problem (3.15) is denoted by ϵ-quasi-$\mathcal{S}(D)$ (resp., ϵ-quasi-$\mathcal{S}^w(D)$).

Theorem 3.1 (Duality) *Let $\bar{x} \in \epsilon$-quasi-$\mathcal{S}^w(P)$ be such that the* (CQ) *defined in* (2.14) *is satisfied at this point. Then there exist $\bar{\lambda} := (\bar{\lambda}_k)$, $\bar{\lambda}_k \geq 0$, $k \in K$, not all zero, and $\bar{\mu} := (\bar{\mu}_i)$, $\bar{\mu}_i \geq 0$, $i \in I$, such that $(\bar{x},\bar{\lambda},\bar{\mu}) \in C_D$ and $f(\bar{x}) = L(\bar{x},\bar{\lambda},\bar{\mu})$. In addition,*
(i) *If f and g are generalized convex on Ω at any $z \in \Omega$, then $(\bar{x},\bar{\lambda},\bar{\mu}) \in \epsilon$-quasi-$\mathcal{S}^w(D)$.*
(ii) *If f is strictly generalized convex and g is generalized convex on Ω at any $z \in \Omega$, then $(\bar{x},\bar{\lambda},\bar{\mu}) \in \epsilon$-quasi-$\mathcal{S}(D)$.*

Theorem 3.2 (Converse Duality) *Let $(\bar{x},\bar{\lambda},\bar{\mu}) \in C_D$ such that $f(\bar{x}) = L(\bar{x},\bar{\lambda},\bar{\mu})$.*
(i) *If $\bar{x} \in C$ and f and g are generalized convex on Ω at $\bar{x}$, then $\bar{x} \in \epsilon$-quasi-$\mathcal{S}^w(P)$.*
(ii) *If $\bar{x} \in C$ and f is strictly generalized convex and g is generalized convex on Ω at $\bar{x}$, then $\bar{x} \in \epsilon$-quasi-$\mathcal{S}(P)$.*

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